

# A compactification of the real configuration space as an operadic completion

Martin Markl

## 1. Introduction and summary.

For a compact Riemannian manifold  $V$ , Axelrod and Singer constructed in [1] a compactification  $C_n(V)$  of the configuration space  $C_n^0(V)$  of  $n$  distinct points in  $V$ , by adding to  $C_n^0(V)$  the blowups along the diagonals. Their construction works also for a noncompact manifold  $V$ . In this case the resulting object will not be compact (the configurations that approach spatial infinity have no limit), so it would perhaps be better to speak about the ‘resolution of diagonals’ rather than about a ‘compactification’, as was done in [7], but we will respect the vicissitudes of history and call the process a ‘compactification’.

There is another, similar compactification of the *moduli space*  $\mathring{F}_m(n)$  of configurations of  $n$  distinct points in the  $m$ -dimensional Euclidean plane  $\mathbf{R}^m$  modulo the action of the affine group, described by Getzler and Jones in [6] and denoted by  $F_m(n)$ . The authors of [6] also stated that the collection  $F_m := \{F_m(n)\}_{n \geq 1}$  has a natural structure of a topological operad. This was a well known fact for  $m = 1$ , because the collection  $F_1 = \{F_1(n)\}_{n \geq 1}$  is nothing else but the operad  $K = \{K_n\}_{n \geq 1}$  of the ‘associahedra’ introduced by J. Stasheff in his work [13] on homotopy associative spaces. Let us remark that for  $m = 2$  the operad  $F_m$  plays an important rôle in topological closed string field theory.

The compactification  $C_n(S^1)$  of the configuration space  $C_n^0(S^1)$  of  $n$  distinct points on the circle was studied by Bott and Taubes in [3] as the basic tool for the construction of ‘nonperturbative’ link invariants. It obviously admits a free  $S^1$ -action and the quotient  $W_n := C_n(S^1)/S^1$  is what J. Stasheff called in [12] the ‘cyclohedron’. In the same paper he observed that the collection  $W := \{W_n\}_{n \geq 1}$  has a natural structure of a right module over the operad  $K = F_1$  in the sense introduced by us in [11, page 1476].

The first aim of this work is to generalize this statement to the case of an arbitrary  $n$ -dimensional Riemannian manifold  $V$ , i.e. to prove that the collection  $C(V) = \{C_n(V)\}_{n \geq 1}$

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has a natural structure of a right module over the operad of the compactification of the moduli space of ‘local configurations’  $F_n$ . Strictly speaking, this is true only for parallelizable manifolds, but even this class contains nontrivial and relevant examples, as we will see later. In the general case we must work with the *framed* version of the compactification, which we introduce in (23). The existence of the above mentioned structures has far-reaching implications to the geometry and combinatorics of the underlying spaces. We will discuss these questions in a forthcoming paper(s), see also the work of M. Ginzburg and A.A. Voronov [9].

We present an entirely new, purely algebraic construction of the compactification based on the fact that configuration spaces have a natural structure of a partial operad (or a partial module over a partial operad, but we will not spoil the picture now). We show that each partial operad admits an ‘operadic completion’ and, by a miracle, this completion shows up to be the compactification we are looking for!

Let us try to give the reader a flavour how this partial operad structure looks. Consider the space  $C_n^0(\mathbf{R}^m)$  of configurations of  $n$  distinct points in the Euclidean plane  $\mathbf{R}^m$ . To define an operad structure on the collection  $C^0(\mathbf{R}^m) = \{C_n^0(\mathbf{R}^m)\}_{n \geq 1}$  we need to specify, for each  $a = (a^1, \dots, a^l) \in C_l^0(\mathbf{R}^m)$  and  $b_i \in C_{m_i}^0(\mathbf{R}^m)$ , the value of the ‘composition map’  $\gamma(a; b_1, \dots, b_l) \in C_{m_1 + \dots + m_l}^0(\mathbf{R}^m)$ . This can be done by putting

$$\gamma(a; b_1, \dots, b_l) := (\underbrace{(a^1, \dots, a^1)}_{m_1 \text{ times}} + b_1, \underbrace{(a^2, \dots, a^2)}_{m_2 \text{ times}} + b_2, \dots, \underbrace{(a_l, \dots, a_l)}_{m_l \text{ times}} + b_l).$$

The configuration  $\gamma(a; b_1, \dots, b_l)$  may be viewed as the superposition of the configurations  $T_{a_1}(b_1), \dots, T_{a_l}(b_l)$ , where  $T_a(-)$  means, just here and now, the translation by a vector  $a \in \mathbf{R}^m$ . This process is visualized on Figure 1.

We encourage the reader to verify that all the axioms of an operad are satisfied. The only small drawback is that  $\gamma(a; b_1, \dots, b_l)$  need not necessarily be an element of the configuration space  $C_{m_1 + \dots + m_l}^0(\mathbf{R}^m)$ , because the components of  $\gamma(a; b_1, \dots, b_l)$  need not be different. Thus the structure map is defined only for *some* elements of  $C_l^0(\mathbf{R}^m) \times C_{m_1}^0(\mathbf{R}^m) \times \dots \times C_{m_l}^0(\mathbf{R}^m)$ ; we will call such an object a *partial operad*, though the definition we use is more subtle and differs a bit from the standard definition of a partial operad.

As far as we know, nobody has observed the existence of this partial operad structure before. It is implicitly hidden in the formulas of [1, pages 25–29], and, in fact, all this paper is based on a very meticulous study of these pages.

configuration  $a = (a^1, a^2)$ :      configuration  $b_1 = (b_1^1, b_1^2, b_1^3)$ :      configuration  $b_2 = (b_2^1, b_2^2)$ :

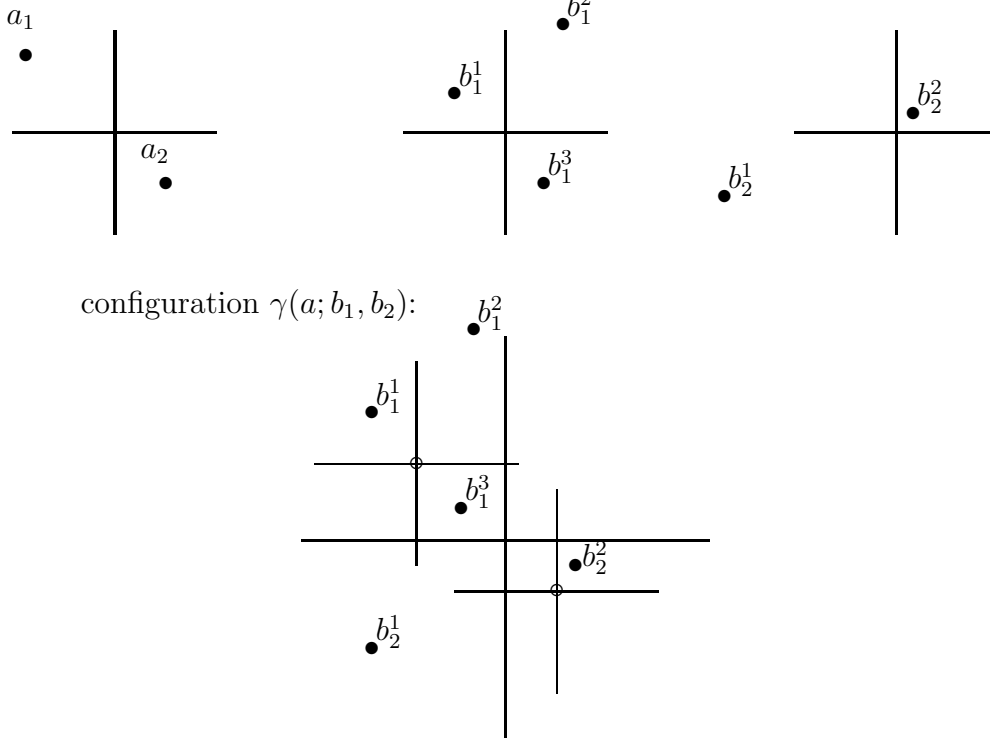


Figure 1: The partial operad structure on the configuration space made easy. The construction of  $\gamma(a; b_1, b_2) \in C_5^0(\mathbf{R}^2)$  from  $a \in C_2^0(\mathbf{R}^2)$ ,  $b_1 \in C_3^0(\mathbf{R}^2)$  and  $b_2 \in C_2^0(\mathbf{R}^2)$ .

As above, we will respect the notations introduced in [6] resp. [3, 15] despite their obvious incompatibility, i.e. we will use the notation  $\overset{\circ}{F}_m(n)$  for the moduli space of configurations of  $n$  distinct points in  $\mathbf{R}^m$  modulo the affine group action (= dilatations and translations), and  $C_n^0(V)$  for the space of configurations of  $n$  distinct points in a manifold  $V$ .

**Summary of the paper.** In the following section we explain our concept of a partial operad and construct an operadic completion of such an object. In Section 3 we define a partial operad of virtual configurations  $\chi$  and a framed version  $f\chi$  of this object. We show that the operadic completion  $\tilde{\chi}$  (resp.  $\widetilde{f\chi}$ ) of  $\chi$  (resp.  $f\chi$ ) coincides with the compactification  $F_m$  (resp. the framed version  $fF_m$ ) considered by Getzler and Jones in [6]. This immediately implies the existence of an operad structure on these objects.

In Section 4 we introduce our notion of partial modules over a partial operad and describe a module completion of these objects. In Section 5 we define, for each Riemannian manifold  $V$ , the partial module of framed virtual configurations  $f\mu$  (resp., if  $V$  is parallelizable, the partial module of virtual configurations  $\mu$ ). We show that the module completion  $\widetilde{f\mu}$  (resp.  $\widetilde{\mu}$ ) coincides with the Axelrod-Singer compactification  $FC(V)$  (resp.  $C(V)$ ). As an immediate consequence we see that  $FC(V)$  is a natural right module over the operad  $fF_m$  (resp. that

$C(V)$  is a natural right module over  $\mathbf{F}_m$ ). Observe that there is many parallelizable manifolds for which the configuration spaces are interesting, for example the spheres  $V = S^m$ , for  $m = 1, 3, 7$ . Another important case is  $V = \mathbf{R}^m$  or  $V =$  the torus, or, still more generally,  $V =$  a (not necessary compact) Lie group.

In the last section we exploit the well-known fact that the above mentioned compactifications are manifold with corners. We get immediately, from a result of J. Cerf [4], that each of those compactifications is diffeomorphic to a closed submanifold obtained by a truncation of its open part. An explicit construction of such a truncation was given, for  $K = \mathbf{F}_1$ , by S. Sternberg and S. Shnider in [14]; the authors show that the associahedron can be constructed as a truncation of the  $(n-1)$  dimensional simplex  $\Delta^{n-1}$ . The possibility of a similar construction of the cyclohedron was observed in the appendix to [12] by J. Stasheff.

For any manifold with corners  $M$ , the skeletal filtration induces a spectral sequence. As suggested by [6, Lemma 3.4], the first term of this spectral sequence can be, for  $M =$  one of the compactifications above, identified to the bar construction (or a suitable generalization) over an operad (or a module) formed by the cohomology of the ‘open parts’ of these spaces. Our approach gives a straightforward definition of these operad structures, more direct than the standard one based on a chain of homotopy equivalences with a little-disks-type object. This gives us a very easy understanding of the first term of this spectral sequence.

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## 2. Algebraic background I.

**2.1. Language of trees.** Let  $\mathcal{T}_n$  denote the set of all (rooted, connected) trees with  $n$  input edges. For such a tree  $T \in \mathcal{T}_n$ , let  $\text{vert}(T)$  denote the set of its vertices. For  $v \in \text{vert}(T)$ , let  $\text{inp}(v)$  be the set of input edges of  $v$ ;  $\text{inp}(v)$  will sometimes denote also the number of input edges of  $v$ , the meaning will always be clear from the context. The set  $\mathcal{T}_n$  of all  $n$ -trees has a natural partial order; we say that  $S \leq T$  if the tree  $S$  was obtained from  $T$  by collapsing one or more of its inner edges. The set  $\mathcal{T}_n$  has a unique minimal element  $T(n)$ , the  $n$ -corolla, the tree with exactly one vertex.

**2.2. Collections and operads.** Recall that a (topological) *collection* is just a sequence  $E = \{E(n)\}_{n \geq 1}$  of topological spaces. For a collection  $E$  and a tree  $T \in \mathcal{T}_n$ , let  $E(T)$  denote the set of all colorings of the vertices of  $T$  by elements of  $E$  such that a vertex  $v$  is colored by an element from  $E(\text{inp}(v))$ ; observe that  $E(T(n)) = E(n)$ . For  $v \in \text{vert}(T)$  and  $\xi \in E(T)$ , denote by  $\xi(v)$  the value of the coloring  $\xi$  at  $v$ . If  $T, S \in \mathcal{T}_n$ ,  $S \leq T$ , then each  $h \in \mathcal{H} := \text{vert}(S)$  labels a subtree  $T_h$  of  $T$  whose vertices collapsed to  $h$ .

For each collection  $E$  there exists the *free operad*  $\mathcal{F}(E)$  generated by  $E$ . As a collection, it is defined by

$$\mathcal{F}(E)(n) = \coprod_{T \in \mathcal{T}_n} E(T),$$

while the operad structure is given by the grafting of the underlying trees. Let us recall [8] that an operad structure on a collection  $E$  can be defined by specifying, for each  $T \in \mathcal{T}_n$ , a structure map  $\gamma_T : E(T) \rightarrow E(T(n)) = E(n)$ ; these maps must behave well under the grafting operation of underlying trees. More precisely, let  $S \leq T \in \mathcal{T}_n$ . Then the restriction defines, for each  $h \in \mathcal{H} := \text{vert}(S)$ , the map  $r_h : E(T) \rightarrow E(T_h)$ . Let us introduce the ‘operadic extension’  $\{\gamma_{S,T}\}_{S \leq T}$  of the system  $\{\gamma_T\}_T$ ,  $\gamma_{S,T} : E(T) \rightarrow E(S)$ , by

$$(1) \quad \gamma_{S,T}(\xi)(h) := \gamma_{T_h}(r_h(\xi)), \quad h \in \text{vert}(S),$$

see 2.1 for the notation. Then we require that

$$(2) \quad \gamma_T(\xi) = \gamma_S(\gamma_{S,T}(\xi)), \quad \text{for each } \xi \in E(T), S, T \in \mathcal{T}_n, S \leq T.$$

**2.3. Partial operads.** We say that a *partial operad* is a collection  $E$  with structure maps defined only on a subset  $U[T]$  of  $E(T)$ ,  $\gamma_T : U[T] \rightarrow E(n)$ ,  $T \in \mathcal{T}_n$ . These maps are supposed to satisfy (2) whenever the corresponding compositions are defined. To understand this better, we introduce the set

$$(3) \quad U_S(T) := \{\xi \in E(T); r_h(\xi) \in U[T_h], h \in \text{vert}(S)\} \subset E(T).$$

Observe that  $U_S(S) = U(S)$ , the set of all colorings of the tree  $S$  by elements of the collection  $U := \{U(n)\}_{n \geq 1}$  with  $U(n) = U[T(n)]$ , while the opposite extreme is  $U_{T(n)}(T) = U[T]$ . The sets  $U_S(S)$  will play the rôle of ‘open strata’ and we denote them by  $R_S$ .

The map  $\gamma_{S,T}$  on the right side of (2) is defined for  $\xi \in U_S(T)$ , we thus require (2) to be satisfied only for

$$(4) \quad \xi \in U[T] \cap U_S(T) \text{ such that } \gamma_{S,T}(\xi) \in U[S].$$

Let  $\mathcal{P} = (E = \{E(n)\}_{n \geq 1}, \{\gamma_T : U[T] \rightarrow E(n)\})$  be a partial operad. We make life easier by assuming that

$$(5) \quad \text{Im}(\gamma_T) \subset U[T(n)] =: U(n), \quad T \in \mathcal{T}_n,$$

as our basic examples will always share this property. Let

$$(6) \quad \hat{U}(T) := \bigcup_{S \leq T} U_S(T),$$

topologized as a subset of  $E(T)$ . Consider the collection  $\hat{U} = \{\hat{U}(n)\}_{n \geq 1}$  defined by

$$\hat{U}(n) := \coprod_{T \in \mathcal{T}_n} \hat{U}(T) \text{ (disjoint union).}$$

**Lemma 2.4.** *The collection  $\hat{U} = \{\hat{U}(n)\}_{n \geq 1}$  is a topological suboperad of the free operad  $\mathcal{F} = \mathcal{F}(E)$ .*

**Proof.** The proof is almost immediate. Let  $\xi \in U_S(T)$ ,  $S \leq T \in \mathcal{T}_l$ , and  $\xi_i \in U_{S_i}(T_i)$ ,  $S_i \leq T_i \in \mathcal{T}_{m_i}$ ,  $1 \leq i \leq l$ . Let  $\gamma(T; T_1, \dots, T_l)$  (resp.  $\gamma(S; S_1, \dots, S_l)$ ) denote the tree obtained by grafting the tree  $T_i$  at the  $i$ -th input of  $T$  (resp. the tree  $S_i$  at the  $i$ -th input of  $S$ ), for  $1 \leq i \leq l$ . Clearly  $\gamma(S; S_1, \dots, S_l) \leq \gamma(T; T_1, \dots, T_l) \in \mathcal{T}_{m_1 + \dots + m_l}$ . If  $\gamma_{\mathcal{F}}$  denotes the composition map of the free operad  $\mathcal{F}$ , we immediately see that

$$\gamma_{\mathcal{F}}(\xi; \xi_1, \dots, \xi_l) \in U_{\gamma(S; S_1, \dots, S_l)}(\gamma(T; T_1, \dots, T_l))$$

which finishes the proof. ■

Condition (5) implies that the map  $\gamma_{S,T}$  introduced in (1) maps  $U_S(T)$  to  $U_S(S) = R_S$ . We may thus define  $\tilde{P} = \{\tilde{P}(n)\}_{n \geq 1}$  by

$$(7) \quad \tilde{P}(n) := \hat{U}(n) / \sim$$

where the relation  $\sim$  identifies elements  $\xi$  of  $U_S(T)$  with their images  $\gamma_{S,T}(\xi) \in R_S \subset E(S)$ . In the following proposition,  $U = \{U(n)\}_{n \geq 1}$  is the collection defined in (5) and  $\mathcal{F}(U)$  is the free operad generated by this collection.

**Proposition 2.5.** *The collection  $\tilde{\mathcal{P}} = \{\tilde{P}(n)\}_{n \geq 1}$  is a topological operad. There exists a natural epimorphism of topological operads*

$$\rho : \mathcal{F}(U) \rightarrow \tilde{\mathcal{P}}.$$

*If the sets  $U_S(T)$  are ‘combinatorially independent’ in the sense that*

$$(8) \quad S', S'' \leq T, \quad S' \neq S'' \implies U_{S'}(T) \cap U_{S''}(T) = \emptyset,$$

*then the map  $\rho$  is an isomorphism of sets.*

In the light of the proposition, we may view  $\tilde{\mathcal{P}}(n)$  as obtained by glueing the ‘open strata’  $U(T) = R_T$ ,  $T \in \mathcal{T}_n$ , of  $\mathcal{F}(U)(n)$  in a way compatible with the operad structure. We call  $\tilde{\mathcal{P}}$  the *operadic completion* of the partial operad  $\mathcal{P}$ .

**Proof of the proposition.** We prove that the operad structure on  $\hat{U}$  induces an operad structure on its quotient (7). To this end, we must show that the equivalence  $\sim$  is compatible with the operad structure on  $\hat{U}$ . This is, however, evident; we defined the system  $\{\gamma_{S,T}\}_{S \leq T \in \mathcal{T}_n}$  by extending  $\{\gamma_T\}_{T \in \mathcal{T}_n}$  as operad maps, and the claim follows from the definition of  $\sim$ .

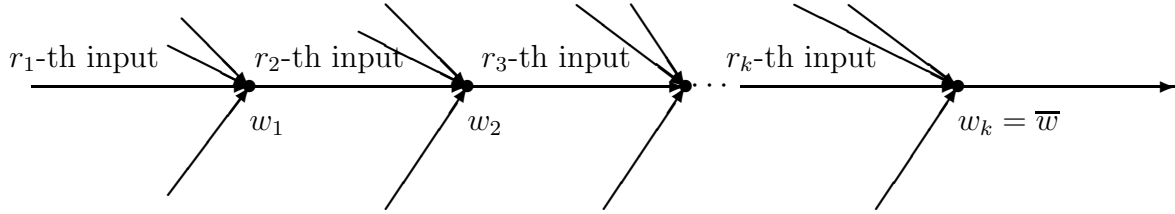
As for the second part of the theorem, the inclusion  $\iota : U \rightarrow \hat{U}$  of collections given by

$$\iota(n) : U(n) = U[T(n)] = U_{T(n)}(T(n)) \hookrightarrow \hat{U}(n)$$

extends to a continuous map  $\rho : \mathcal{F}(U) \rightarrow \tilde{\mathcal{P}}$ , by the freeness of the operad  $\mathcal{F}(E)$ . The very definition of the relation  $\sim$  implies that each  $\xi \in \hat{U}(n)$  is equivalent to some  $\xi' \in R_S \subset \text{Im}(\rho)$ . This implies that the map  $\rho$  is an epimorphism. The independence condition (8) then assures that the relation  $\sim$  cannot identify two distinct points of  $R_S$ , which shows that  $\rho$  is a monomorphism. ■

### 3. Compactification of the moduli space.

**3.1.** We open this section by defining the *partial operad of virtual configurations*  $\chi = (E, \{\gamma_T : U[T] \rightarrow E(n)\})$ . The collection  $E$  is given by  $E(n) := [\mathbf{R}_{\geq 0} \times \mathring{\mathbf{F}}_m(n)]$  for  $n \geq 2$ , while  $E(1) = \emptyset$ . We must also specify, for each  $T \in \mathcal{T}_n$ , a subset  $U[T] \subset E(T)$  and a composition map  $\gamma_T : U[T] \rightarrow E(n)$ . Since  $E(1) = \emptyset$ , the set  $E(T)$  can be nonempty only

Figure 2: A path in the tree  $T$ .

for trees all of whose vertices have at least two input edges; we denote the set of all such trees by  $\mathcal{T}_n^{\geq 2}$ .

First of all, an element of  $E(T)$  is a sequence

$$(9) \quad \xi = \{\kappa_w; w \in \mathcal{W}\}, \quad \kappa_w = (t_w, [\vec{z}_w]) \in [\mathbf{R}_{\geq 0} \times \mathring{\mathbf{F}}_m(\text{inp}(w))] \text{ for } w \in \mathcal{W} = \text{vert}(T).$$

We can assume that the vectors  $\vec{z}_w = (z_w^1, \dots, z_w^{i_w})$ , where  $i_w := \text{inp}(w)$ , are *normalized* in the sense that

$$(10) \quad \sum_{1 \leq i \leq i_w} z_w^i = 0 \text{ and } \sum_{1 \leq i \leq i_w} |z_w^i|^2 = 1,$$

where  $|\cdot|$  denotes the Euclidean norm in  $\mathbf{R}^m$ .

Let  $\overline{w}$  be the terminal vertex of the tree  $T$  and let  $Y(T) \subset E(T)$  be the set of all elements as in (9) such that  $t_{\overline{w}} = 0$ . As the first step towards  $\gamma_T$  we define, for  $T \in \mathcal{T}_n^{\geq 2}$ , a map  $\omega_T : Y(T) \rightarrow (\mathbf{R}^m)^n$  as follows.

For any  $1 \leq i \leq n$  there exists in  $T$  a unique path from the  $i$ -th input to the output, as in Figure 2. Using the notation above, we put

$$(11) \quad \omega_i(\xi) := z_{\overline{w}}^{r_k} + t_{w_{k-1}} \cdot z_{w_{k-1}}^{r_{k-1}} + \dots + t_{w_1} \dots t_{w_{k-1}} \cdot z_{w_1}^{r_1}$$

and, finally,  $\omega_T(\xi) := (\omega_1(\xi), \dots, \omega_n(\xi))$ . The following observation is interesting and we formulate it though we will not need it in the sequel; the proof is immediate.

**Observation 3.2.** *The map  $\omega_T : Y(T) \rightarrow (\mathbf{R}_{\geq 0}^m)^n$  is a monomorphism, for any  $T \in \mathcal{T}_n^{\geq 2}$ .*

Let  $U[T]$  be the set of all  $\xi \in Y(T)$  such that all the points  $\omega_1(\xi), \dots, \omega_n(\xi) \in \mathbf{R}^m$  are distinct. Then  $\gamma_T : U[T] \rightarrow E(n)$  is defined as the composition of the restriction  $\omega_T|_{U[T]}$  with the projection  $C_n^0(\mathbf{R}^m) \rightarrow \mathring{\mathbf{F}}_m(n)$  and the inclusion  $\mathring{\mathbf{F}}_m(n) = \{0\} \times \mathring{\mathbf{F}}_m(n) \hookrightarrow E(n)$ .



**Proposition 3.3.** *The object*

$$\chi = (E = \{E(n)\}_{n \geq 1}, \{\gamma_T : U[T] \rightarrow E(n)\}_{T \in \mathcal{T}_n^{\geq 2}})$$

*defined above is a partial operad satisfying the independence condition (8).*

**Proof.** Let us prove the combinatorial independence first. If  $\xi \in U_S(T)$  is as in (9) then, by definition,  $r_h(\xi) \in U[T_h]$  for all  $h \in \mathcal{H}$ , see (3). This clearly implies that  $t_w = 0$  if and only if  $w$  is the output vertex of some  $T_h$ . Thus the set  $\{w \in \text{vert}(T); t_w = 0\}$  uniquely determines a tree  $S$  with  $S \leq T$  such that  $\xi \in U_S(T)$ ; the combinatorial independence is now obvious.

We must of course verify also that  $\chi$  is a partial operad. But this is easy: the independence implies that, if  $U_S(T) \cap U[T] \neq \emptyset$ , then  $S = T(n)$ , the  $n$ -corolla. Thus, by (4), the only thing which has to be verified is the unitarity,  $\gamma_{T(n)} = \text{id}$ , which is immediate from the definition. ■

**Theorem 3.4.** *The operad completion  $\tilde{\chi}$  of the partial operad  $\chi$  coincides with the compactification  $F_m$  of the moduli space of points in the plane discussed in [6],  $F_m(n) = \tilde{\chi}(n)$  for any  $n \geq 1$ .*

**Proof.** We prove the theorem by constructing an explicit isomorphism  $z : \tilde{\chi}(n) \rightarrow F_m(n)$ . Let  $\xi = \{\kappa_w = (t_w, [\vec{z}_w]); w \in \mathcal{W}\} \in U_S(T) \subset E(T)$  be a point as in (9). As we already saw in the proof of Proposition 3.3, the tree  $S$  uniquely determines a subset  $\mathcal{W}_S \subset \mathcal{W}$ ;  $\mathcal{W}_S := \{w \in \mathcal{W}; t_w = 0\}$ . For each  $\epsilon > 0$  define  $\tau_\xi(\epsilon) \in E(T)$  by

$$\tau_\xi(\epsilon) = \{\lambda_w = (s_w, [\vec{z}_w]); w \in \mathcal{W}\},$$

where  $s_w := t_w$  for  $w \in \mathcal{W} \setminus \mathcal{W}_S$ , and  $s_w := \epsilon$  for  $w \in \mathcal{W}_S$ . By [1, lemma in §5.4],  $\tau_\xi(\epsilon) \in U[T]$  for small  $\epsilon$ . Thus, for small  $\epsilon$ ,  $\gamma_T(\tau_\xi(\epsilon))$  is a curve in  $\mathring{F}_m(n)$  which converges, for  $\epsilon \rightarrow 0$ , to a point in the compactification  $F_m(n)$ . We define

$$z(\xi) := \lim_{\epsilon \rightarrow 0} (\gamma_T(\tau_\xi(\epsilon))) \text{ in } F_m(n).$$

We must prove that this definition is compatible with the defining relation  $\sim$  of (7). This was in fact done in the proof of a theorem in [1, §5.4], and our claim becomes clear if we compare our  $\gamma_{S,T}$  with the formulas [1, (5.77.1), (5.82)] for the extension of the map  $\psi_0$ , though the verification is rather difficult because of the difference between notations used.

Fortunately, the claim can be verified more or less directly, if we realize what we are suppose to verify.

We have to verify the following. If  $\xi$  is as above, let  $\xi' := \gamma_{S,T}(\xi) \in U_S(S)$  and let  $\tau_{\xi'}(\epsilon) \in E(S)$  be the corresponding curve. Then we must prove that

$$\lim_{\epsilon \rightarrow 0}(\gamma_T(\tau_{\xi}(\epsilon))) = \lim_{\epsilon \rightarrow 0}(\gamma_S(\tau_{\xi'}(\epsilon))) \text{ in } F_m(n).$$

To do this, we must write explicit formulas for the curves  $\gamma_T(\tau_{\xi}(\epsilon))$  and  $\gamma_S(\tau_{\xi'}(\epsilon))$  and then use a criterion of [1, §5.2] to compare points in the compactification which are presented as limits of curves in the ‘open part’. This is a straightforward, though not exactly easy, verification. ■

**3.5.** We need also a ‘framed’ version of the operad  $\overset{\circ}{F}_m$ . It will be an ‘ $G$ -operad’ with  $G = \mathbf{O}(n)$ , where by a  $G$ -operad we mean an operad  $\mathcal{P} = \{\mathcal{P}(n)\}_{n \geq 1}$  such that each  $\mathcal{P}(n)$  is a (left)  $G$ -space and the composition map satisfies

$$(12) \quad g(\gamma(x; x_1, \dots, x_l)) = \gamma(gx; x_1, \dots, x_l), \quad x \in \mathcal{P}(l), \quad x_i \in \mathcal{P}(m_i), \quad 1 \leq i \leq l, \quad g \in G.$$

A typical example of such an object is the  $\mathbf{O}(m)$ -operad  $f\mathcal{D}_m$  of framed little  $m$ -disks. More generally, suppose we have an (ordinary) operad  $\mathcal{P}$  such that each  $\mathcal{P}(n)$  is a (left)  $G$ -space and such that the composition map satisfies, under the notation of (12),  $g\gamma(x; x_1, \dots, x_l) = \gamma(gx; gx_1, \dots, gx_l)$ . An example is the ordinary little  $m$ -disks operad  $\mathcal{D}_m$  with the action of  $\mathbf{O}(m)$  induced by the representation of this group on the ambient affine space. Then the operad  $G\mathcal{P}$  with  $G\mathcal{P}(n) := \mathcal{P}(n) \times G^{\times n}$ , with the diagonal action of the group  $G$  and the composition map  $\gamma_G$  defined as

$$\begin{aligned} \gamma_G((x, g_1, \dots, g_l); (x_1, g_1^1, \dots, g_1^{m_1}), \dots, (x_l, g_l^1, \dots, g_l^{m_l})) := \\ (\gamma(gx; x_1, \dots, x_l), g_1 g_1^1, \dots, g_1 g_1^{m_1}, \dots, g_l g_l^1, \dots, g_l g_l^{m_l}) \end{aligned}$$

is a  $G$ -operad in the sense of (12). We believe that the analogous notion of a *partial*  $G$ -operad is clear.

We are going to define now, for any  $m \geq 1$ , the partial  $\mathbf{O}(m)$ -operad of framed virtual configurations  $f\chi = (fE, \{f\gamma_T : fU[T] \rightarrow fE(n)\})$ . Let  $fE(n) := E(n) \times \mathbf{O}(n)^{\times n}$ , where

the collection  $E(n)$  is the same as in the definition of the partial operad  $\chi$  in 3.1. A typical element  $\rho \in fE(T)$ ,  $T \in \mathcal{T}_n^{\geq 2}$ , looks like

$$\rho = \{\rho_w; w \in \mathcal{W}\}, \rho_w = (t_w, [\vec{z}_w], \vec{g}_w), w \in \mathcal{W}, \text{ with} \\ t_w \in \mathbf{R}_{\geq 0}, [\vec{z}_w] = [z_w^1, \dots, z_w^{i_w}] \in \mathring{\mathbf{F}}_m(i_w), \vec{g}_w = (g_w^1, \dots, g_w^{i_w}) \in \mathbf{O}(m)^{\times i_w}, i_w := \text{inp}(w).$$

Let  $fY = \{\rho \in fE(T); t_{\bar{w}} = 0\}$ , where  $\bar{w}$  is the output vertex of  $T$ . For  $1 \leq i \leq n$ , we define the framed version of the map  $\omega_i$  of (11) as

$$(13) \quad f\omega_i(\phi) := z_{\bar{w}}^{r_k} + t_{w_{k-1}} \cdot (g_{w_{k-1}}^{r_{k-1}} z_{w_{k-1}}^{r_{k-1}}) + \dots + t_{w_1} \cdots t_{w_{k-1}} \cdot (g_{w_{k-1}}^{r_{k-1}} \cdots g_{w_1}^{r_1} z_{w_1}^{r_1}),$$

where we use the same notation based on Figure 2 as in (11). As before, put

$$fU[T] = \{\rho \in Y(T); \text{ the points } f\omega_1(\rho), \dots, f\omega_n(\rho) \text{ are distinct}\}.$$

Finally, let  $f\gamma_T(\rho) := [\omega(\rho)] \times (g_1, \dots, g_n)$ , where  $[\omega(\rho)]$  denotes the class of  $\omega(\rho) \in C_n^0(\mathbf{R}^m)$  in  $\mathring{\mathbf{F}}_m(n)$  and  $g_i := g_{w_k}^{r_k} \cdots g_{w_1}^{r_1}$ ,  $1 \leq i \leq n$ . We have the following ‘framed’ version of Proposition 3.3.

**Proposition 3.6.** *The object*

$$f\chi = (fE = \{fE(n)\}_{n \geq 1}, \{f\gamma_T : fU[T] \rightarrow fE(n)\}_{T \in \mathcal{T}_n^{\geq 2}})$$

*defined above is a partial  $\mathbf{O}(m)$ -operad satisfying the independence condition (8). It contains  $\chi$  as a natural suboperad.*

It is well-known [6] that the compactification  $\mathbf{F}_m(n)$  of the moduli space  $\mathring{\mathbf{F}}_m(n)$  admits a natural action of the group  $\mathbf{O}(m)$ . This action can be used to introduce the ‘framed’ version  $f\mathbf{F}_m(n)$  of the space  $\mathbf{F}_m(n)$  by  $f\mathbf{F}_m(n) := \mathbf{F}_m(n) \times \mathbf{O}(m)^{\times n}$ , with the diagonal action of the group  $\mathbf{O}(n)$ . We have the following analog of Theorem 3.4.

**Theorem 3.7.** *The operadic completion  $\widetilde{f\chi}$  of the partial  $\mathbf{O}(m)$ -operad  $f\chi$  coincides with the framed version  $f\mathbf{F}_m$  of the compactification  $\mathbf{F}_m$  of the moduli space of points in the plane introduced above,  $f\mathbf{F}_m(n) = \widetilde{f\chi}(n)$  for any  $n \geq 1$ .*

#### 4. Algebraic background II.

**4.1. Modules over operads.** Let  $M$  and  $E$  be topological collections and  $T \in \mathcal{T}_n$  a tree. Denote by  $M_E(T)$  the set of all colorings of the tree  $T$  such that the output vertex of  $T$  is colored by an element of  $M$  while the remaining vertices are colored by elements of  $E$ . Suppose that the collection  $E$  forms an operad with the structure maps  $\{\gamma_T : E(T) \rightarrow E(n)\}$  as in 2.2. One way to define on a collection  $M$  a *right module structure* over the operad  $E$  in the sense of [11] is to specify maps  $\nu_T : M_E(T) \rightarrow M(n)$ ,  $T \in \mathcal{T}_n$ , which behave well in the following sense, compare 2.2.

Let  $T, S \in \mathcal{T}_n$ ,  $S \leq T$ . Let  $\mathcal{H} := \text{vert}(S)$  and let  $\bar{h}$  be the output vertex of the tree  $S$ . Decompose  $\mathcal{H}$  as  $\mathcal{H} = \{\bar{h}\} \cup \mathcal{H}'$ . The restriction gives the map  $r_{\bar{h}} : M_E(T) \rightarrow M(T_{\bar{h}})$  and, for each  $g \in \mathcal{H}'$ , the map  $r_g : M_E(T) \mapsto E(T_g)$ . We define the ‘modular extension’  $\{\nu_{S,T}\}_{S \leq T}$  of the system  $\{\nu_T\}_T$ ,  $\nu_{S,T} : M_E(T) \rightarrow M_E(S)$ , by

$$(14) \quad \nu_{S,T}(\eta)(\bar{h}) := \nu_{T_{\bar{h}}}(\eta)(\bar{h}) \text{ and } \nu_{S,T}(\eta)(g) := \gamma_{T_g}(r_g(\eta)), \quad g \in \mathcal{H}'.$$

Then we require that

$$(15) \quad \nu_T(\eta) = \nu_S(\nu_{S,T}(\eta)), \text{ for } \eta \in M_E(T), S \leq T, S, T \in \mathcal{T}_n.$$

**4.2. Partial modules.** Let  $\mathcal{P} = (E, \gamma_T : U[T] \rightarrow E(n))$  be a partial operad as in 2.3. Then a structure of a *partial module* over a partial operad  $\mathcal{P}$  will be given by specifying, for each  $T \in \mathcal{T}_n$ , a subset  $W[T] \subset M_E(T)$  and a map  $\nu_T : W[T] \rightarrow M(n)$  such that the maps  $\{\nu_T\}_T$  satisfy (15), whenever the compositions involved are defined. As in 2.3 this means that we require (15) only for  $\eta \in W[T] \cap W_S(T)$  with  $\nu_{S,T}(\eta) \in W[S]$ , where  $W_S(T)$  is the subset of  $M_E(T)$  defined as

$$(16) \quad W_S(T) := \{\eta \in M_E(T); r_{\bar{h}}(\eta) \in W[T_{\bar{h}}] \text{ and } r_g(\eta) \in U[T_g], \quad g \in \mathcal{H}'\}.$$

We suppose, as in (5), that

$$(17) \quad \text{Im}(\nu_T) \subset W[T(n)] =: W(n), \quad T \in \mathcal{T}_n.$$

Let  $\mathcal{M} = (M = \{M(n)\}_{n \geq 1}, \{\nu_T : W[T] \rightarrow M(n)\})$  be a partial right module over a partial operad  $\mathcal{P} = (P = \{E(n)\}_{n \geq 1}, \{\gamma_T : U[T] \rightarrow E(n)\})$ . The constructions which we introduced for partial operads in Section 2 carry over almost literally. For  $S \leq T$ ,  $S, T \in \mathcal{T}_n$ , put

$$(18) \quad \hat{W}(T) := \bigcup_{S \leq T} W_S(T) \subset M_E(T).$$

As in the proof of Lemma 2.4 we may show that the collection  $\hat{W} = \{\hat{W}(n)\}_{n \geq 1}$  defined by

$$\hat{W}(n) := \coprod_{T \in \mathcal{T}_n} \hat{W}(T)$$

is a topological submodule of the free right module  $M \circ \mathcal{F}(E)$  generated by the collection  $M$  over the free operad  $\mathcal{F} = \mathcal{F}(E)$  (a strange notation is justified by [10]).

As in the case of partial operads, condition (17) assures that formula (14) defines the map  $\nu_{S,T} : W_S(T) \rightarrow W_S(S) =: S_S$ . Let  $\tilde{\mathcal{M}} = \{\tilde{\mathcal{M}}(n)\}_{n \geq 1}$  be given by

$$(19) \quad \tilde{\mathcal{M}}(n) := \hat{W}(n) / \sim$$

with the relation  $\sim$  identifying elements  $\eta$  of  $W_S(T)$  with their images  $\nu_{S,T}(\eta) \in S_S \subset M_E(S)$ .

In the following proposition,  $W = \{W(n)\}_{n \geq 1}$  is the collection defined in (17),  $U = \{U(n)\}_{n \geq 1}$  is the collection with  $U(n) = U[T(n)]$ ,  $n \geq 1$  (compare (5)), and  $W \circ \mathcal{F}(U)$  is the free right  $\mathcal{F}(U)$ -module generated by the collection  $W$ .

**Proposition 4.3.** *The collection  $\tilde{\mathcal{M}} = \{\tilde{\mathcal{M}}(n)\}_{n \geq 1}$  is a topological right module over the operadic completion  $\tilde{\mathcal{P}}$  of the partial operad  $\mathcal{P}$ . There exists a natural epimorphism of topological right modules*

$$\delta : W \circ \mathcal{F}(U) \rightarrow \tilde{\mathcal{M}}.$$

*If the sets  $W_S(T)$  are ‘combinatorially independent’ in the sense that*

$$(20) \quad S', S'' \leq T, \quad S' \neq S'' \implies W_{S'}(T) \cap W_{S''}(T) = \emptyset,$$

*then the map  $\delta$  is an isomorphism of sets.*

The proof is essentially identical to the proof of Proposition 2.5. We call  $\tilde{\mathcal{M}}$  the *module completion* of the partial right module  $\mathcal{M}$ .

## 5. Compactification of configuration spaces.

Let  $V$  be an  $m$ -dimensional Riemannian manifold. Recall that  $C_n^0(V)$  denotes the space of configurations of  $n$  distinct points in  $V$ . This space has a straightforward ‘framed’ version

$$FC_n^0(V) := \{\vec{x} \times (f_1, \dots, f_n); \quad \vec{x} = (x_1, \dots, x_n) \in C_n^0(V), \quad f_i \in F_{x_i}(V), 1 \leq i \leq n\},$$

where  $F(V)$  is the principal  $\mathbf{O}(m)$ -bundle of frames on the manifold  $V$ . Thus  $FC_n^0(V)$  is the space of configurations of  $n$  distinct points of  $V$ , each decorated with a frame. Another, fancier, way is to define  $FC_n^0(V)$  as the pullback of the product bundle  $F(V)^{\times n} \rightarrow V^n$  under the inclusion  $C_n^0(V) \hookrightarrow V^n$ .

If the tangent bundle of the manifold  $V$  is trivial, then the trivialization defines an isomorphism  $FC_n^0(V) \cong C_n^0(V) \times [\mathbf{O}(m)]^{\times n}$ , which induces the inclusion

$$(21) \quad C_n^0(V) = C_n^0(V) \times [\mathbb{1}]^m \hookrightarrow FC_n^0(V) \quad (\mathbb{1} \text{ is the unit of } \mathbf{O}(m)).$$

We will define the partial right module  $\mu = (M = \{M(n)\}_{n \geq 1}, \{\nu_T : W[T] \rightarrow W(n)\})$  (resp.  $f\mu = (fM = \{fM(n)\}_{n \geq 1}, \{\nu_T : fW[T] \rightarrow fW(n)\})$ ) of (framed) virtual configurations of points in the manifold  $V$ , over the partial operad  $\chi$  of virtual configurations (resp. over the partial  $\mathbf{O}(m)$ -operad  $f\chi$  of framed virtual configurations) of points in  $\mathbf{R}^m$ . Let us start with the definition of  $f\mu$ .

The collection  $fM$  is simply  $fM := FC^0(V)$ . The definition of the subsets  $fW[T] \subset fM_{fE}(T)$  is more difficult. Observe first that, if  $fM_{fE}(T) \neq \emptyset$ , then all vertices of the tree  $T$ , except maybe the output one, have at least two input edges; we denote the set of all such  $n$ -trees by  $\mathcal{T}_n^{\geq 2,e}$ . Let  $\mathcal{V} := \text{vert}(T)$ ,  $\mathcal{V} = \{\bar{v}\} \cup \mathcal{V}'$ , where  $\bar{v}$  is the output vertex of the tree  $T$ . A typical element of  $fM_{fE}(T)$  can be written as  $\eta = \{\lambda_v; v \in \mathcal{V}\}$ , with

$$(22) \quad \begin{aligned} \eta &= \{\lambda_v; v \in \mathcal{V}\}, \lambda_{\bar{v}} = \vec{x} \times \vec{f}, \quad \text{where } \vec{x} = (x_1, \dots, x_l) \in C_l^0(V), \\ \vec{f} &= (f_1, \dots, f_l), f_i \in F_{x_i}(V), l = \text{inp}(\bar{v}), \text{ and} \\ \lambda_u &= (t_u, [\vec{z}_u], \vec{g}_u), t_u \in \mathbf{R}_{\geq 0}, [\vec{z}_u] = [z_u^1, \dots, z_u^{i_u}] \in \mathring{F}_m(i_u), \text{ and} \\ \vec{g}_u &= (g_u^1, \dots, g_u^{i_u}) \in \mathbf{O}(m)^{\times i_u}, i_u := \text{inp}(u), \text{ for } u \in \mathcal{V}'. \end{aligned}$$

For each vertex  $u \in \mathcal{V}'$  there is a unique path in  $T$  joining  $u$  and  $\bar{v}$  as in Figure 2 (with  $u$  instead of  $w_1$ ,  $u_2$  instead of  $w_2, \dots, \bar{v}$  instead of  $\bar{w}$ ). Put  $x_u := x_{r_k}$  and  $f_u := g_{u_{k-1}}^{r_{k-1}} \cdots g_u^{r_1} \cdot f_{r_k}$ . The frame  $f_u$  identifies  $\mathbf{R}^m$  with the tangent space  $T_{x_u}(V)$  of the manifold  $V$  at the point  $x_u$ , so we may suppose that  $[\vec{z}_u]$  is an element of  $C_{\text{inp}(u)}^0(T_{x_u}(V))/\text{Aff}$ . We may moreover suppose that  $\vec{z}_u = (z_u^1, \dots, z_u^{i_u}) \in C_{i_u}^0(T_{x_u})$ , is *normalized* in the sense that

$$\sum_{1 \leq i \leq i_u} z_u^i = 0 \text{ and } \sum_{1 \leq i \leq i_u} |z_u^i|^2 = 1,$$

where  $|\cdot|$  denotes the norm induced by the Riemannian metric.

For  $1 \leq i \leq n$  there exists in  $T$  a unique path from the  $i$ -th input to the output, as in Figure 2 (with  $u_1$  instead of  $w_1, \dots, \bar{v}$  instead of  $\bar{w}$ ). Then put

$$\varphi_i(\eta) := \exp_{x_{r_k}}(t_{u_{k-1}} \cdot z_{u_{k-1}}^{r_{k-1}} + \dots + t_{u_1} \dots t_{u_{k-1}} \cdot z_{u_1}^{r_1}), \quad \varphi(\eta) := (\varphi_1(\eta), \dots, \varphi_n(\eta)) \in V^n.$$

It might seem strange, when we compare this formula to (13), that the coefficient at  $z_{u_j}^{r_j}$  does not contain the product  $g_{w_{i-1}}^{r_{i-1}} \dots g_{w_1}^{r_1}$ . This is because this expression is already a part of the identification of  $\mathbf{R}^m$  to the tangent space  $T_{x_u}(V)$ .

For  $x \in V$ ,  $v \in T_x(V)$  and  $z := \exp_x(v)$  define the ‘parallel transport’  $\Phi_{x,z} : T_x(V) \rightarrow T_z(V)$  by

$$\Phi_{x,z}(w) := \left. \frac{d}{dt} \right|_{t=0} \exp_x(v + tw), \quad w \in T_x(V),$$

compare [1, (5.80)]. Define  $\Phi(\vec{f}) := (\Phi(f_1), \dots, \Phi(f_n))$ , where  $\Phi(f_i) := \Phi_{x_{r_k}, \varphi_i(\eta)}(f_i) \in T_{\varphi_i(\eta)}(V)$ .

Then  $fW[T]$  is the set of all  $\eta \in fM_E(T)$  such that the points  $\varphi_1(\xi), \dots, \varphi_n(\xi) \in V$  are distinct. The structure map  $\nu_T : fW[T] \rightarrow fM(n)$  is defined as  $\nu_T(\eta) := \varphi(\eta) \times \Phi(\vec{f})$ , for  $\eta \in fW[T]$ .

As we already observed in (21), if the tangent bundle of the manifold  $V$  is trivial, then the collection  $M := C^0(V)$  is a subcollection of the collection  $fM = FC_n^0(V)$  and, of course,  $E$  is a subcollection of  $fE$ . Thus we may put  $W[T] := fW[T] \cap M_E(T)$ . We may moreover suppose that the Riemannian metrics on  $V$  is induced by the trivialization. This means that the ‘parallel transport’  $\Phi$  leaves the subcollection  $C^0(V)$  of  $FC^0(V)$  invariant and  $\nu_T$  restricts to a map (denoted by the same symbol)  $\nu_T : W[T] \rightarrow W(n)$ .

**Proposition 5.1.** *The object*

$$f\mu = (fM = \{fM(n)\}_{n \geq 1}, \{\nu_T : fW[T] \rightarrow fW(n)\}_{T \in \mathcal{T}_n^{\geq 2, e}})$$

*is a partial right module over the partial operad  $f\chi$  of framed virtual configurations and thus also over the partial suboperad  $\chi \subset f\chi$ . It satisfies the independence condition (20).*

*If  $V$  is parallelizable, then the object*

$$\mu = (M = \{M(n)\}_{n \geq 1}, \{\nu_T : W[T] \rightarrow W(n)\}_{T \in \mathcal{T}_n^{\geq 2, e}})$$

*is a partial  $\chi$ -submodule of the partial module  $f\mu$ .*

There is an obvious framed version of the Axelrod-Singer compactification  $C_n(V)$ . Let  $\pi = (\pi_1, \dots, \pi_n) : C_n(V) \rightarrow V^n$  be the ‘blow down’ map, then put

$$(23) \quad FC_n(V) := \{\xi \times (f_1, \dots, f_n); \xi \in C_n(V), f_i \in F_{\pi_i(\xi)}(V), 1 \leq i \leq n\}.$$

As in (21), if  $V$  is parallelizable, then  $C_n(V)$  is a natural subspace of  $FC_n(V)$  for all  $n \geq 1$ . Now we may formulate the main theorem of this section.

**Theorem 5.2.** *The module completion  $\widetilde{f\mu}$  of the partial module  $f\mu$  coincides with the framed version of the Axelrod-Singer compactification  $FC(V)$ ,  $FC_n(V) = \widetilde{f\mu}(n)$  for any  $n \geq 1$ . This implies, among other things, that  $FC(V)$  is a natural right  $fF_m$ -module.*

*If  $V$  is parallelizable, then the module completion  $\widetilde{\mu}$  of the partial module  $\mu$  is a natural right topological  $F_m$ -submodule of  $\widetilde{f\mu}$ . It coincides with the Axelrod-Singer compactification  $C(V)$  of the moduli space of points in  $V$ ,  $C_n(V) = \widetilde{\mu}(n)$  for any  $n \geq 1$ .*

As we have already observed in Proposition 5.1, we may also consider  $f\mu$  as a partial right module *over*  $\chi$ . One can expect that the module completion of  $f\mu$  as a partial module *over*  $\chi$  will be smaller than the completion over  $f\mu$ . We leave to the reader the proof of the following proposition, see (23) for the notation.

**Proposition 5.3.** *The module completion of  $f\mu$  as a partial module over  $\chi$  consists of all elements  $\xi \times (f_1, \dots, f_n) \in FC_n(V)$  such that  $f_i = f_j$  whenever  $\pi_i(\xi) = \pi_j(\xi)$ ,  $1 \leq i, j \leq n$ .*

## 6. Manifolds-with-corners and spectral sequences.

It is well-known that the spaces  $F_m(n)$  and  $C_n(V)$  are manifolds with corners. Since  $fF_m(n) = F_m(n) \times \mathbf{O}(m)^{\times n}$  and, at least ‘locally’,  $FC_n(V) \cong C_n(V) \times \mathbf{O}(m)^{\times n}$ , also the framed versions have structures of a manifold with corners. We get immediately from [4, Proposition 1, page 257] the following proposition which says, roughly speaking, that the compactifications discussed above are ‘truncations’ of their open parts.

**Proposition 6.1.** *Each of the spaces  $F_m(n)$ ,  $fF_m(n)$ ,  $C_n(V)$  and  $FC_n(V)$  is isomorphic to a closed submanifold (with corners) of its open part  $\overset{\circ}{F}_m(n)$ ,  $\overset{\circ}{fF}_m(n)$ ,  $C_n^0(V)$  and  $FC_n^0(V)$ , respectively, obtained by removing a collar neighborhood of the boundary.*



We need, however, a deeper and more explicit understanding of these structures. Recall that for a partial operad  $\mathcal{P}$  and a tree  $T \in \mathcal{T}_n$  we introduced the ‘open stratum’  $R_T = U_T(T)$  and, similarly, for a partial right module  $M$  over  $\mathcal{P}$  we have the ‘open strata’  $S_T = W_T(T)$ . Let  $\mathcal{T}_n(p)$  be the subset of  $\mathcal{T}_n$  consisting of trees with exactly  $(p - 1)$  vertices.

**Lemma 6.2.** *Let  $\mathcal{P} = \chi$  or  $f\chi$ , then for each  $T \in \mathcal{T}_n(p)$  there exists a ‘collar neighborhood’  $\mathcal{N}(T)$  of the stratum  $R_T$  in  $\hat{U}(T)$ , isomorphic to  $R_T \times (\mathbf{R}_{\geq 0})^p$ .*

*Similarly, for  $M = \mu$  or  $f\mu$ , there exists a ‘collar neighborhood’  $\mathcal{N}(T)$  of the stratum  $S_T$  in  $\hat{W}(T)$ , isomorphic to  $S_T \times (\mathbf{R}_{\geq 0})^p$ .*

**Proof.** Let us prove the lemma for  $\mathcal{P} = \chi$ , the proof of the remaining three cases is analogous. Fix a tree  $T \in \mathcal{T}_n^{\geq 2}$  and let  $\xi \in Y(T)$  be as in (9), i.e.

$$(24) \quad \xi = \{\kappa_w; w \in \mathcal{W}\}, \quad \kappa_w = (t_w, [\vec{z}_w]) \in [\mathbf{R}_{\geq 0} \times \mathring{\mathbf{F}}_m(\text{inp}(w))], \quad t_{\bar{w}} = 0,$$

with  $\mathcal{W} = \{\bar{w}\} \cup \mathcal{W}' = \text{vert}(T)$ , where  $\bar{w}$  is the output vertex of  $T$ . We claim that for any  $\phi = \{[\vec{z}_w]; w \in \mathcal{W}\} \in R_T = \mathring{\mathbf{F}}_m(T)$  there exists an  $\epsilon_\phi > 0$  such that, if  $t_u < \epsilon_\phi$  for all  $u \in \mathcal{W}'$ , then the element  $\xi$  of (24) lies in  $U(T)$ . This follows from the usual continuity argument and the observation that if all  $t_u$ ’s are ‘almost’ zero, then certainly  $\xi \in U(T)$ . Then the set  $\mathcal{N}(T) = \{\xi; t_u \leq \epsilon_\phi, \phi \in \mathring{\mathbf{F}}_m(T)\}$  obviously has the required property. ■

Consider the collection  $\mathring{\mathbf{F}}_m = \{\mathring{\mathbf{F}}_m(n)\}_{n \geq 1}$  and the associated homology collection  $\mathbf{e}_m := \{\mathbf{e}_m(n)\}_{n \geq 1}$  in the category of graded vector spaces given by  $\mathbf{e}_m(n) := H_*(\mathring{\mathbf{F}}_m(n))$ . This collection is well-known to have a natural structure of an operad. A traditional way to show this fact is first to observe that  $\mathring{\mathbf{F}}_m(n)$  is homotopically equivalent to  $C_n^0(\mathbf{R}^m)$  (because  $\mathring{\mathbf{F}}_m(n) = C_n^0(\mathbf{R}^m)/\text{Aff}$  and the group  $\text{Aff}$  is contractible) while the latter space is homotopically equivalent to  $\mathcal{D}_m(n)$ , the  $n$ -th piece of the little disk operad.

The system of collar neighborhoods of Lemma 6.2 however defines this operad structure in a straightforward way. Since we obviously have  $H_*(\mathcal{N}(T)) = H_*(\mathring{\mathbf{F}}_m(T)) = \mathbf{e}_m(T)$ , the restriction  $\gamma_T|_{\mathcal{N}(T)} : \mathcal{N}(T) \rightarrow U(n) = \mathring{\mathbf{F}}_m(n)$  induces, for each  $T \in \mathcal{T}_n^{\geq 2}$ , a map  $\gamma_T^{\mathbf{e}} : \mathbf{e}_m(T) \rightarrow \mathbf{e}_m(n)$ . We leave to the reader the verification of the following proposition.

**Proposition 6.3.** *The system  $\{\gamma_T^{\mathbf{e}} : \mathbf{e}_m(T) \rightarrow \mathbf{e}_m(n)\}$  defines an operad structure on the collection  $\mathbf{e}_m$ . This structure coincides with the structure induced by the little  $m$ -disks operad as explained above, i.e. the operad  $\mathbf{e}_m$  describes  $n$ -algebras in the sense of [6].*

Similarly, the partial operad  $f\chi$  of framed virtual configurations induces an operad structure on the collection  $f\mathbf{e}_m := H_*(f\mathring{\mathbf{F}}_m)$ , this operad describes  $m$ -dimensional analogs of Batalin-Vilkovisky algebras, compare [5].

Analogous principle applies to the collections  $f\mathbf{m}(V) := H_*(FC^0(V))$  (and  $\mathbf{m}(V) := H_*(C^0(V))$  if  $V$  is parallelizable). As above we have the following proposition.

**Proposition 6.4.** *The partial  $f\chi$ -module  $f\mu = (fM, \{\nu_T : fW[T] \rightarrow fW(n)\})$  of virtual configurations of points in  $V$  induces on the collection  $f\mathbf{m}(V)$  a structure of a right module over the operad  $f\mathbf{e}_m$ .*

*If  $V$  is parallelizable, then there is an analogous right  $\mathbf{e}_m$ -module structure on the collection  $\mathbf{m}(V)$  induced by the partial  $\chi$ -module  $\mu = (M, \{\nu_T : W[T] \rightarrow W(n)\})$ .*

For an  $n$ -dimensional manifold with corners  $M$ , denote by  $M[p]$  the union of the faces of  $M$  with codimension  $p$ , and by  $F_p M$  its closure (sometimes called the *codimension  $p$  skeleton*). Recall Lemma 3.3 of [6] (but, since we do not assume  $M$  to be compact, we must work with the cohomology with compact supports).

**Lemma 6.5.** *The filtration  $F_p M$  induces a spectral sequence with  $E_{pq}^1 = H_q(M[p])$  converging to  $H_{\text{comp}}^{n-*}(M)$ . The differential  $d^1 : H_q(M[p]) \rightarrow H_q(M[p-1])$  is identified, by the Lefschetz duality, with the boundary map  $\delta$  of the cohomology exact sequence of the triple  $(F_{p-1}M, F_p M, F_{p+1}M)$ .*

The main theorem of this section uses the notion of the bar construction over an operad. This notion has already become a standard one, we thus only briefly recall the definition and refer the reader to [8, 6] for details.

Let  $\mathcal{Q}$  be an operad in the category of graded vector spaces and denote by  $\uparrow\mathcal{Q}$  the suspension of the collection  $\mathcal{Q}$ , i.e.  $\uparrow\mathcal{Q} = \{(\uparrow\mathcal{Q})(n)\}_{n \geq 1}$ , where  $(\uparrow\mathcal{Q})(n) := \uparrow\mathcal{Q}(n)$  is the ordinary suspension of the graded vector space  $\mathcal{Q}(n)$ . The *bar construction on the operad  $\mathcal{Q}$*  is the differential collection  $\mathcal{B}(\mathcal{Q}) = \{\mathcal{B}(\mathcal{Q})(n)\}_{n \geq 1}$  with

$$\mathcal{B}(\mathcal{Q})(n) := \bigoplus_{T \in \mathcal{T}_n^{\geq 2}} (\uparrow\mathcal{Q})(T)$$

and let the differential  $d_{\mathcal{B}} : \mathcal{B}(\mathcal{Q})(n) \rightarrow \mathcal{B}(\mathcal{Q})(n)$  of degree  $-1$  defined as follows. Let  $T \in \mathcal{T}_n^{\geq 2}$  and let  $e \in \text{edg}(T)$  be an inner edge. If we denote by  $T/e \in \mathcal{T}_n^{\geq 2}$  the tree

obtained by collapsing the edge  $e$ , then the operad composition on  $\mathcal{Q}$  clearly defines a map  $\delta_{T,T/e} : (\uparrow \mathcal{Q})(T) \rightarrow (\uparrow \mathcal{Q})(T/e)$ . The differential is then given by

$$d_{\mathcal{B}}(x) = \sum_{e \in \text{edg}(T)} \pm \delta_{T,T/e}(x), \text{ for } x \in (\uparrow \mathcal{Q})(T).$$

The sign is a tricky part here. It is determined by demanding  $d_{\mathcal{B}}$  to be a degree  $-1$  coderivation of the cofree cooperad  $\mathcal{B}(\mathcal{Q})$ . We do not need the exact formula for the sign here, so we just refer the reader to the above mentioned sources [8, 6] for details. Similarly, let  $M$  be a right  $\mathcal{Q}$ -module. Let us define the *bar resolution*  $\mathcal{B}(M, \mathcal{Q}) = \{\mathcal{B}(M, \mathcal{Q})(n)\}_{n \geq 1}$  of the right  $\mathcal{Q}$ -module  $M$  by

$$\mathcal{B}(M, \mathcal{Q})(n) = \bigoplus_{T \in \mathcal{T}_n^{\geq 2, e}} (\uparrow M)_{(\uparrow \mathcal{Q})}(T)$$

with the differential  $d_{\mathcal{B}} : \mathcal{B}(M, \mathcal{Q})(n) \rightarrow \mathcal{B}(M, \mathcal{Q})(n)$  is defined analogically as the differential of the bar construction. Compare also [9, 2]. Again, the bar resolution  $\mathcal{B}(M, \mathcal{Q})$  can be shown to be a right comodule over the cooperad  $\mathcal{B}(\mathcal{Q})$ . The case  $M = \mathbf{F}_m$  of the following proposition was proven in [6, Lemma 3.3].

**Theorem 6.6.** *In the spectral sequence for the manifold with corners  $M = \mathbf{F}_m(n)$ , the term  $(E^1, d^1)$  is naturally isomorphic to the  $n$ -th piece of the bar construction  $\mathcal{B}(\mathbf{e}_m)$  on the operad  $\mathbf{e}_m$ ,  $(E^1, d^1) \cong (\mathcal{B}(\mathbf{e}_m)(n), d_{\mathcal{B}})$ . Similarly, for  $M = f\mathbf{F}_m(n)$ , the first term is isomorphic to the  $n$ -th piece of  $\mathcal{B}(f\mathbf{e}_m)$ .*

*For  $M = FC_n(V)$ , the first term  $(E^1, d^1)$  is isomorphic to  $(\mathcal{B}(f\mathbf{m}_m, f\mathbf{e}_m)(n), d_{\mathcal{B}})$ . If the manifold  $V$  is parallellizable, then, for  $M = C_n(V)$ ,  $(E^1, d^1) \cong (\mathcal{B}(\mathbf{m}_m, \mathbf{e}_m)(n), d_{\mathcal{B}})$ .*

**Proof.** Standard calculations show that the map  $d^1 : H_q(M[p]) \rightarrow H_q(M[p-1])$  is induced by the inclusion  $M[p] \subset M[p-1]$  which, of course, does not exist in the literal sense. We must first thicken  $M[p]$ , considered as a part of the boundary of  $F_{p-1}M$ , into a collar neighborhood and then move it a bit into the interior of this neighborhood, which is a subset of  $M[p-1]$ .

We will describe this process in details for  $M = \mathbf{F}_m(n)$  where the notation is easiest, but all the remaining cases can be discussed in exactly the same way. We know that the set  $M[p]$  is the disjoint union of the open strata  $R_T = \mathring{\mathbf{F}}_m(T)$  over all trees  $T \in \mathcal{T}_n^{\geq 2}(p)$ . Similarly,  $M[p-1] = \coprod \{\mathring{\mathbf{F}}_m(S); S \in \mathcal{T}_n^{\geq 2}(p-1)\}$ . It is clear from the construction that  $R_T$  may intersect the closure of  $R_S$  in  $\mathbf{F}_m(n)$  if and only if  $S = T/e$ , for some inner edge  $e \in \text{edg}(T)$ .

Let  $\phi = \{[\vec{z}_w]; w \in \mathcal{W} = \text{vert}(T)\} \in S_T$  and let  $w_0$  be the input vertex of  $e$ . As in the proof of Lemma 6.2 there exists  $\epsilon_\phi > 0$  such that

$$\{(t_w, [\vec{z}_w]); w \in W, t_w = 0 \text{ for } w \neq w_0, t_{w_0} < \epsilon_\phi\} \subset U_S(T).$$

We may moreover suppose that  $\epsilon_\phi$  depends continuously on  $\phi$ . Then

$$\tilde{S}_T := \{(t_w, [\vec{z}_w]); w \in W, t_w = 0 \text{ for } w \neq w_0, t_{w_0} = \tfrac{1}{2}\epsilon_\phi\}.$$

is an isomorphic copy of  $S_T$  in  $U_S(T)$ . The corresponding component of the differential  $d^1$  is then induced by the composition  $S_T \cong \tilde{S}_T \subset U_S(T) \xrightarrow{\gamma_{S,T}} S_S \subset M[p-1]$ . Our description of the operad structure then immediately identifies this map to  $\delta_{T,T/e}$ . ■

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Mathematical institute of the Academy, Žitná 25, 115 67 Prague 1, The Czech Republic  
e-mail: `markl@mbox.cesnet.cz`